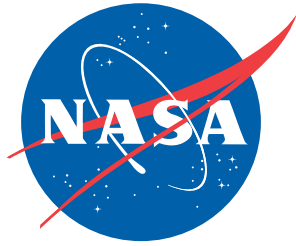


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Probabilistic Requirements (Partial) Verification Methods Best Practices Improvement

*Variables Acceptance Sampling Calculators: Derivations
and Verification of Plans*

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February 2012

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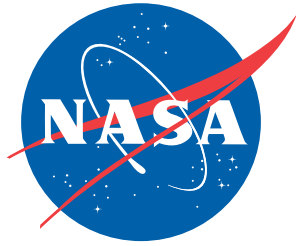
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February 2012

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October 27, 2011

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Approval and Document Revision History

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Version	Description of Revision	Office of Primary Responsibility	Effective Date
1.0	Initial Release	Kenneth L. Johnson, Statistics and Trending, Marshall Space Flight Center	10/27/11


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Volume I: Technical Assessment Report

1.0 Notification and Authorization


The NASA Engineering and Safety Center (NESC) was requested to improve on the Best Practices document produced for the NESC assessment 07-001-E, Verification of Probabilistic Requirements for the Constellation Program (CxP), by giving a recommended procedure for using acceptance sampling by variables techniques. This recommended procedure would be used as an alternative to the potentially resource-intensive acceptance sampling by attributes method given in the document.

An NESC out-of-board activity was approved on January 9, 2008. Mr. Kenneth Johnson at Marshall Space Flight Center (MSFC) was selected to lead this assessment. The assessment plan was approved at the NESC Review Board (NRB) on March 20, 2008.

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3.0 Team List

Name	Discipline	Organization/Location
Core Team		
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Chris Johansen	MTSO	LaRC
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4.0 Abstract

Acceptance sampling is a method for verifying quality or performance requirements using sample data. Acceptance sampling by variables (ASV) is an alternative to acceptance sampling by attributes (ASA). In many instances, ASV requires significantly smaller samples than ASA. In an effort to make ASV more widely accessible, the team developed calculators for existing sampling plans identified in the open literature. For this paper, the literature on ASV was consolidated by providing a unified exposition of the approach used to develop such plans. From within this framework, the derivation of plans for exponential, normal, gamma, Weibull, inverse-Gaussian, and Poisson random variables were reviewed and verified. Verification unexpectedly surfaced a flaw in a published plan for inverse-Gaussian variables. A companion paper confirmed these results empirically and addressed the practical application of ASV when distributional forms were themselves uncertain.

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5.0 Introduction

One of the oldest problems in quality engineering is to assess the acceptability of items that a customer receives from a producer. Acceptance sampling is an alternative to 100-percent inspection applied when inspection is destructive, or when the time and/or cost of 100-percent inspection are unwarranted or prohibitive. The customer decides the disposition of an incoming lot based on a standard specifying the minimum proportion of nonconforming items in the sample. The decision can be to accept or reject the entire lot, or to continue sampling.


Acceptance sampling also has been adapted to problems not identified with procurement. Smith *et al.* (2003) advocated the application of acceptance sampling in the context of water quality assessment. Bayard *et al.* (2007) applied the underlying concept to the assessment of future aircraft runway-incursion controls. White, *et al.* (2009) showed that acceptance sampling can be applied to any sampling experiment, including those commonly employed to verify design requirements using simulation and Monte Carlo methods.

ASA determines the acceptability of a lot based on a count of the number of nonconforming items relative to the size of a random sample drawn from this lot. The inspection variable is binary (pass/fail) and therefore the count is necessarily a binomial random variable. ASA can be used with categorical outputs, or with outputs measured on a continuous or discrete scale, by reference to a required limiting value. Conceptually simple, easily applied, and universally applicable, ASA is the first choice for sampling inspection.

ASV is an alternate approach, which in many instances prescribes significantly smaller samples than ASA. ASV requires that the inspection variable is measured on a continuous or discrete scale, that the distribution of this variable is known *a priori* and stable, and that a plan exists for this particular distribution. While far more restrictive in its assumptions, ASV may be considered when the assumptions are appropriate and the larger samples required by ASA are unavailable.

White, *et al.* (2009) describes the development of ASA plans and provides an example illustrating the application of ASV to a Level 2 requirement from the Constellation Program (CxP). The first objective of this paper was to consolidate the scholarly literature on ASV and provide a readable tutorial on the concept and general approach used to develop ASV plans for alternative parent distributions. While ASV does employ a consistent procedure for determining sampling plans, this commonality was obscured by different presentation styles and the vastly different notation adopted by researchers in the field. This paper presents a unified presentation and a consistent notation. An example is provided in Appendix A.

The second objective was to validate the mathematical derivations. Validation was successful in all but one case. A subtle flaw in the published plan for inverse-Gaussian variables was discovered that rendered it unusable, at least in its present form. A companion paper confirmed

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these results and addressed the practical application of ASV when distributional forms were themselves uncertain.

6.0 Probabilistic Requirements and Limit Standards


In uncertain environments, requirements verification seeks to determine whether a measurable quantity is conforming or nonconforming, i.e., to determine whether or not the parent population from which a sample is drawn achieves a specified level of quality or performance. It is important to remember that this is a simplified and incomplete view of verification. A true requirement verification using these methods alone would be incomplete—a requirement can be judged as verified when a full understanding of the assumptions is clearly presented and understood. NASA-STD-7009, *Standard for Models and Simulations* (2008), offers a framework for doing much of this. The discussion on epistemic versus aleatory uncertainties in the *Probabilistic Risk Assessment Procedures Guide for NASA Managers and Practitioners* Version 1.1 (2002)¹ is also recommended. It is important to recognize that this verification process does not generally apply at all to the “Probable Risk Assessment-type” requirements that directly address loss of crew or vehicle.

The remainder of the document will address this partial verification. The methods outlined in this paper were meant to improve the rigor of the portion of the requirements verification process involving the raw exercise of a simulation model and the comparison of its output to the stated quantified requirement threshold. These methods do not address the critical wider issue of analysis method assumptions.

A probabilistic requirement can be stated probabilistically as a (I, ρ, β) limit standard (White, *et al.*, 2009), where:

- (1) I is the *performance indicator*. The measured quantity may be inherently categorical or qualitative in nature and performance indicated by occurrence or nonoccurrence of some event. Alternately, the measured quantity may be inherently quantitative and performance is indicated by success or failure in achieving a limit or tolerance.
- (2) ρ is the *minimum reliability* for the population. This is the minimum, acceptable, long-run proportion of observations on which the population achieves the desired performance. If p is the *failure probability* for the population, then the requirement is $p \leq 1 - \rho$.

¹ Probabilistic Risk Assessment Procedures Guide for NASA Managers and Practitioners Version 1.1 has been replaced by NASA/SP-2011-3421 Probabilistic Risk Assessment Procedures Guide for NASA Managers and Practitioners second edition.

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- (3) β is the *maximum acceptable consumer's risk*. This is the probability of incorrectly accepting a nonconforming population as the result of sampling instead of 100-percent inspection.

Formally, a measurable quantity is represented as the random variable X with a probability distribution of $F(x; \theta)$ and density is $f(x; \theta)$, where θ is an unknown distribution parameter. Let $\{X_i; i = 1, \dots, n\}$ be a random sample with observed values $\{x_i; i = 1, \dots, n\}$. The verification problem is to determine whether the population as a whole conforms to a specified limit standard, based on the statistics of the sample observed.

7.0 Sampling Plans and Operating Characteristics

A *sampling plan* is the pair (n, κ) , where n is the minimum sample size, i.e., the minimum number of observations required to verify statistically the requirement imposed by the standard. κ is a constant factor which is used to assess whether or not the population is conforming. The interpretation of κ depends on whether the characteristic is continuous or discrete, as discussed in Section 8.0.

For a given distribution, every sampling plan has a unique operating characteristic (OC). The OC is a function that defines the probability of accepting a population, P_a , for every value of the failure probability, p , i.e., the OC is the function $P_a(p)$ where $p \in [0, 1]$. A sampling plan is derived by first defining two *operating points*, $(p_0, 1 - \alpha)$ and (p_1, β) , where $p_0 < p_1$ and α and β are small probabilities. It is then required that OC must (minimally) satisfy the following inequalities:

$$P_a(p_0) \geq 1 - \alpha \quad (\text{Eq. 1})$$

$$P_a(p_1) \leq \beta. \quad (\text{Eq. 2})$$

Alternately, it is sometimes more convenient to express these in terms of the rejection probabilities as the power conditions $P_r(p_0) = 1 - P_a(p_0) \leq \alpha$ and $P_r(p_1) = 1 - P_a(p_1) \geq 1 - \beta$. Note that when $p_1 = 1 - \rho$ and β is the consumer's risk, inequality (Eq. 2) enforces the limit standard. Under this plan, a population with failure probability p_1 as conforming to small probability β is noted. Inequality (Eq. 1) captures the competing good. Under this plan, a population with probability p_0 as conforming to high probability $1 - \alpha$ is noted.

The probability of incorrectly rejecting a conforming system, α , was called the producer's risk. When given as percentages, 100-percent α ($1 - p_0$) was called the acceptable quality level and 100-percent β ($1 - p_1$) was called the lot tolerance percent defective. Note that the limit standard does not specify the $(p_0, 1 - \alpha)$ operating point. It was common to develop a range of plans with the required consumer's risk β and differing α . Plans with larger sample sizes n then have smaller corresponding α .

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8.0 Hypothesis Testing and Acceptance Limits

The underlying problem was framed as a hypothesis test, which was intended to enforce both significance and power requirements. The null and alternate hypotheses are:

$$H_0: p = p_0 \text{ and } H_1: p = p_1 > p_0$$

Under H_0 , the population was accepted as conforming and under H_1 the population was rejected as nonconforming. Inequality (Eq. 1) establishes the significance of test as α and inequality (Eq. 2) establishes the power of the test as $1 - \beta$, where α and β are the probabilities of a Type I error (the producer's risk or the risk of rejecting a population that should have been accepted), and a Type II error (the consumer's risk, or the risk of accepting a population that should have been rejected), respectively.

The sample data were used to choose between the null and alternate hypotheses. To accomplish this, the critical value of an appropriate test statistic needed to be determined. Denote the test statistic as the *acceptance limit* $A(n, \kappa)$, where the arguments are the parameters of the sampling plan. For a continuous quantity, $\kappa = k$ is the acceptance limit and typically has the form:

$$A(n, \pm k) = \hat{\theta}(n) \pm k \hat{\sigma}(n)$$

where the plus is used with an upper bound and the minus with a lower bound. It follows that:

$$k = \frac{A(n, \kappa) - \hat{\theta}(n)}{\hat{\sigma}(n)} \quad (\text{Eq. 3})$$


where $\hat{\theta}$ and $\hat{\sigma}$ are estimators for the unknown parameter and the standard deviation of the population.

For I a *lower specification limit* with value x_{\min} , the desired performance on the i^{th} observation is achieved if and only if $x_i \geq x_{\min}$. For the sample as a whole, the null hypothesis is rejected if $A(n, -k) \leq x_{\min}$. That is, the acceptance limit is smaller than the specified limit.

For I an *upper specification limit* with value x_{\max} , the desired performance on the i^{th} observation is achieved if and only if $x_i \leq x_{\max}$. For the sample as a whole, the null hypothesis is rejected if $A(n, -k) \geq x_{\max}$. That is, the acceptance limit is larger than the specified limit.

The effect in either case is to move the critical point of the acceptance limit away from the estimate of the unknown quantity in the direction of the specified limit by k standard deviations. Intuitively, this hedge is intended to compensate for error in the estimated mean for a sample of n observations in a statistically exact way.

Note that in this paper the third case will not be addressed, where two limits are specified. In this case, I is a *tolerance interval*, $x_{\max} \geq x_i \geq x_{\min}$. The derivation of sampling plans for tolerance

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is modestly more complicated than that described in Section 8.0. The reader is referred to the literature on tolerance intervals.

For a discrete population characteristic, the acceptance limit has the form:

$$A(n, c) = Y(n) - c$$

where Y is some function of X with nonnegative integer values. It follows that:

$$c = Y(n) - A(n, c). \quad (\text{Eq. 4})$$


For a lower specification limit on Y , the *acceptance number*, c , is the minimum acceptable value of $Y(n)$ and the rejection criteria is $A(n, c) < 0$. For an upper specification limit on Y , c is the maximum acceptable value of $Y(n)$ and the rejection criteria is $A(n, c) > 0$.

9.0 General Procedure for Developing Variables Acceptance Sampling Plans

For a continuous distribution with lower specification limit, the procedure comprises 4 steps:

- (1) Determine the value of θ_0 (typically the mean) for the null probability distribution with failure probability $p_0 = \Pr[X \leq x_{min}] = F(x_{min}; \theta_0)$; determine the value of θ_1 for the alternate distribution with failure probability $p_1 = \Pr[X \leq x_{min}] = F(x_{min}; \theta_1)$. This is accomplished using the inverse distribution. As shown in Figure 9.0-1, the requirement that $p_0 < p_1$ implies that $\theta_0 > \theta_1$.
- (2) Determine the minimum value of n from the sampling distributions for θ_0 and θ_1 such that both inequalities (Eq. 1) and (Eq. 2) are satisfied.
- (3) Determine the maximum acceptance limit, A_0 , from the null sampling distribution with $\alpha = \Pr[\hat{\theta} \leq A_0] = F(\hat{\theta}; \theta_0, n)$; determine the minimum acceptance limit, A_1 , from the alternate sampling distribution with $\beta = \Pr[\hat{\theta} \leq A_1] = F(\hat{\theta}; \theta_1, n)$; as illustrated in Figure 9.0-2.
- (4) Determine the factors k_0 and k_1 corresponding to A_0 and A_1 , respectively, from equation (3). In general, these factors are not equal. A conservative choice is to use the larger of the two values as the factor k for the sampling plan. Alternately, k is sometimes taken as the average of these two values.

For I , an upper specification limit, the failure probability of the population is $p = \Pr[X > x_{max}] = 1 - F(x; \theta)$. The procedure is identical with this single exception. For a discrete distribution, instead of the factors k_0 and k_1 , at Step 4 the constant c is determined.

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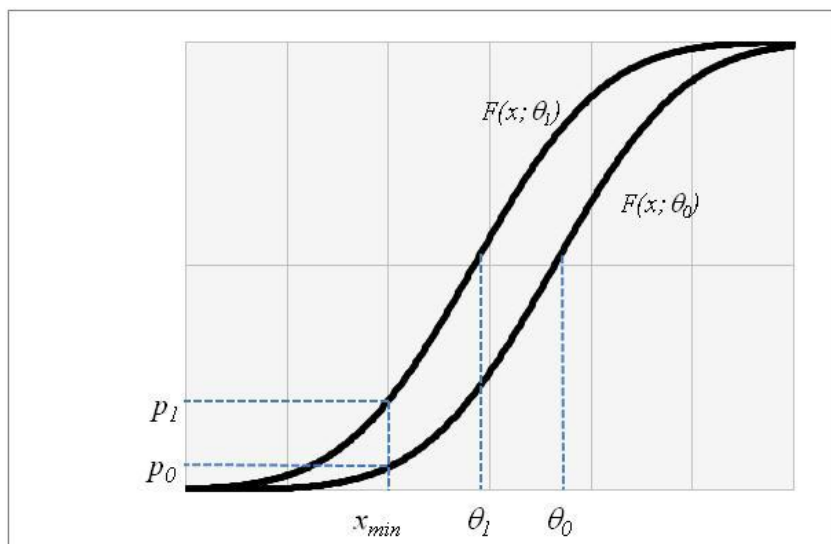


Figure 9.0-1. Distribution Functions and Critical Values for X Under the Null and Alternate Hypotheses

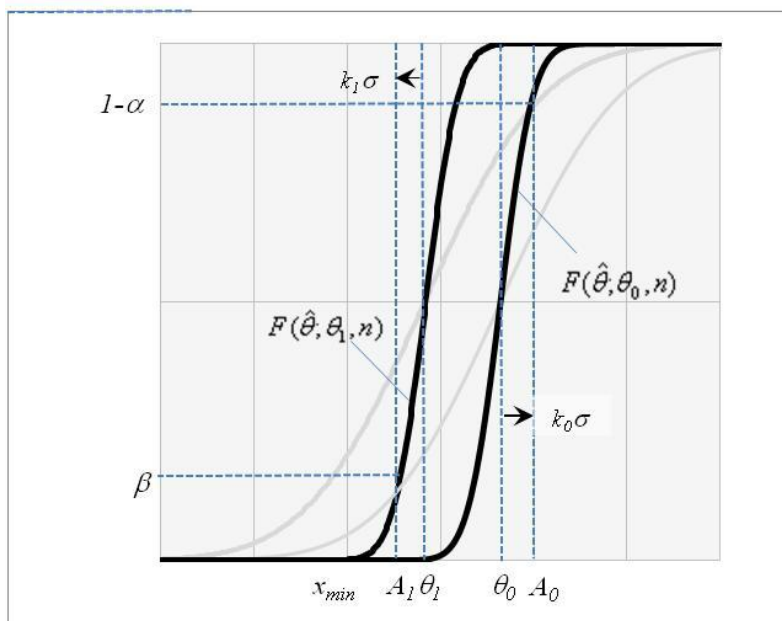


Figure 9.0-2. Sampling Distributions for the Estimator $\hat{\theta}$ Under the Null and Alternate Hypotheses

If the random variable X and the unknown parameter θ can be standardized, it is generally more convenient to use the standardized distribution and standardized sampling distribution to derive

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sampling plans. This convenience is illustrated in Section 10 for exponential and normal random variables.

10.0 Methods Implemented

A literature search found derivations of (n,k) sampling plans for exponential, normal, gamma, Weibull, and inverse-Gaussian distributions. These derivations follow the general procedure given in Section 9.0. A derivation was also found for the (n,c) sampling plan for the Poisson distribution. In this section, these derivations are outlined with reference to lower specification limits. The corresponding derivations for upper specification limits are easily deduced.

10.1 Exponential

Guenther (1977) considers the exponential random variable X with unknown mean μ , supports $x \in [0, \infty)$, and distribution function:

$$F(x; \mu) = 1 - e^{-x/\mu}.$$

Note that exponential and chi-squared distributions are both special cases of the gamma distribution, with exponential $(\mu) = \text{chi-squared}(2) = \text{gamma}(1, \mu)$.

Denote a random variable distributed chi-squared with ν degrees of freedom as Y_ν . X can be standardized as:

$$Y_2 = \frac{2}{\mu} X$$

so that $p = \Pr(X \leq x_p) = \Pr(Y_\nu \leq y_{2,p})$ and

$$y_{2,p} = \frac{2}{\mu} x_{\min} \quad (\text{Eq. 5})$$


where p is either p_0 or p_1 . Similarly, the estimator for the mean μ is sample mean $\bar{X}(n)$, which can be standardized as:

$$Y_{2n} = \frac{2n}{\mu} \bar{X}(n). \quad (\text{Eq. 6})$$

Substituting equation (5) into equation (6) yields:

$$y_{2n;p} = \frac{2n}{\mu} \bar{x}(n) = \frac{ny_{n;p}}{x_{\min}} \bar{x}(n)$$

where p is either p_0 or p_1 . Inequalities (Eq. 1) and (Eq. 2) imply:

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$$y_{2n;p_0} = \frac{2n}{\mu} \bar{x}(n) = \frac{y_{2;p_0}}{x_{\min}} \bar{x}(n) \leq y_{2n;\alpha}$$

$$y_{2n;p_1} = \frac{2n}{\mu} \bar{x}(n) = \frac{y_{2;p_1}}{x_{\min}} \bar{x}(n) \geq y_{2n;(1-\beta)}.$$

Together, these in turn imply that:

$$\frac{y_{2n;\alpha}}{y_{2;p_0}} \geq \frac{\bar{x}(n)}{x_{\min}} \geq \frac{y_{2n;(1-\beta)}}{y_{2;p_1}}$$

or

$$\frac{y_{2;p_1}}{y_{2;p_0}} \geq \frac{y_{2n;(1-\beta)}}{y_{2n;\alpha}}. \quad (\text{Eq. 7})$$

The minimum required sample size is the smallest value of the integer n satisfying inequality (Eq. 7), which can be determined by table lookup.

With n known, the k -factor can be computed. The estimator for both the mean and standard deviation is the sample mean, $\bar{X}(n)$; therefore, the acceptance limit is:

$$A(n, -k) = \hat{\mu}(n) - k' \hat{\sigma}(n) = (1-k) \bar{X}(n) = (1-k') \frac{\mu}{2n} Y_{2n}.$$

It is customary to let $k = 1 - k'$ and use the k -factors and

$$k_0 = \frac{ny_{2;p_0}}{y_{2n;\alpha}}$$

$$k_1 = \frac{ny_{2;p_1}}{y_{2n;(1-\beta)}}.$$

10.2 Normal (σ -known)

The derivation of sampling plans for measureable quantities distributed $N(\mu, \sigma^2)$ is widely published (Bowker and Goode, 1952; Guenther, 1977; Kao, 1971; Lieberman and Resnikoff, 1955; and Montgomery, 2005; among others) and the basis for standards Military Standard (MIL-STD)-414 and American National Standards Institute/American Society of Quality Control Z1.9, and International Organization for Standardization 2859. Consider the

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normal random variable X with unknown mean μ , known standard deviation σ , supports $x \in (-\infty, \infty)$, and density function:

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}.$$

Denote a random variable distributed a standard normal $N(0,1)$ as Z . X can be standardized as:

$$Z = \frac{X - \mu}{\sigma}$$

such that $p = \Pr(X \leq x_p) = \Pr(Z \leq z_p)$ is:

$$z_p = \frac{A - \bar{x}(n)}{\sigma}. \quad (\text{Eq. 8})$$

The mean estimator $\bar{X}(n)$ is distributed $N(\mu, \sigma^2/n)$, which can be standardized as:

$$\bar{Z}(n) = \frac{\bar{X}(n) - \mu}{\sigma/\sqrt{n}} \quad (\text{Eq. 9})$$

where $\bar{Z}(n)$ also is a standard normal $N(0,1)$ deviate.

The acceptance criterion is $A(n, -k) = \bar{x}(n) - k\sigma \geq x_{\min}$, so that the population is rejected if:


$$\frac{\bar{x}(n) - x_{\min}}{\sigma} < k.$$

Inequalities (Eq. 1) and (Eq. 2) imply:

$$\begin{aligned} \Pr\left[\frac{\bar{X}(n) - x_{\min}}{\sigma} < k\right] &= \Pr\left[\frac{\bar{X}(n) - x_{\min}}{\sigma/\sqrt{n}} < k\sqrt{n}\right] \\ &= \Pr\left[\frac{\bar{X}(n) - \mu}{\sigma/\sqrt{n}} + \frac{\mu - x_{\min}}{\sigma/\sqrt{n}} < k\sqrt{n}\right] \\ &= \Pr\left[\bar{Z} - z_p\sqrt{n} < k\sqrt{n}\right] \\ &= \Pr\left[\bar{Z} < \sqrt{n}(k + z_p)\right]. \end{aligned}$$

The acceptance-limit criteria (expressed as power requirements) from inequalities (Eq. 1) and (Eq. 2) then imply:

$$\sqrt{n}(k + z_{p_0}) \geq z_\alpha$$

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$$\sqrt{n}(k + z_{p_1}) \geq z_{(1-\beta)} = z_{1-\beta} = -z_{\beta}.$$

Together, these imply that:

$$\begin{aligned} k_0 &= \frac{z_{\alpha}}{\sqrt{n}} k - z_{p_0} \\ k_1 &= \frac{-z_{\beta}}{\sqrt{n}} - z_{p_1} \end{aligned} \quad (\text{Eq. 10})$$

and therefore

$$n \geq \left[\frac{z_{\alpha} + z_{\beta}}{z_{p_1} - z_{p_0}} \right]^2. \quad (\text{Eq. 11})$$

The minimum required sample size is the smallest integer value of n satisfying inequality (Eq. 11). With n known, the k -factors can be computed as from equations (Eq. 10).

10.3 Normal (σ -unknown)

The more usual case is where σ is unknown and must be estimated from the sample data. While an exact approach is available using a non-central t -distribution (see Guenther, 1977, among others). For modest sample sizes, excellent approximations for n and k are achieved applying a result by Cramér (1945). Let $\bar{X}(n)$ and $S(n)$ be sample mean and standard deviation, respectively. For n sufficiently large, the distribution of the acceptance limit $A(n, k) = \bar{X}(n) - kS(n)$ is asymptotically normally distributed $N(\mu_A, \sigma_A^2)$, with mean


$$\mu_A = \mu \pm k\sigma$$

and variance:

$$\sigma_A^2 = \frac{\sigma^2}{n} e(k, \xi_3, \xi_4).$$

The limiting variance σ^2/n is weighted by the *expansion factor* $e(k, \xi_3, \xi_4)$ (Takagi, 1972). For a *non-Normal* random variable X with mean μ , standard deviation σ , skew ξ_3 , and kurtosis ξ_4 ,

$$e(k, \xi_3, \xi_4) = \left[1 + \frac{k^2}{4} (\xi_4 - 1) \pm k\xi_3 \right] \quad (\text{Eq. 12})$$

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where the plus is used with an upper bound and the minus with a lower bound. For a *Normal* random variable X , the skew is $\xi_3 = 0$ and the kurtosis is $\xi_4 = 3$. The expansion factor therefore reduces to (Wallis, 1947):

$$e(k, 0, 3) = \left[1 + \frac{k^2}{2} \right].$$

The derivation of n and k then proceeds as in the σ -known case, resulting in:

$$n \geq e(k, 0, 3) \left[\frac{z_\alpha + z_\beta}{z_{p_1} - z_{p_0}} \right]^2$$

and

$$k = - \left[\frac{z_\alpha z_{p_1} + z_\beta z_{p_0}}{z_\alpha + z_\beta} \right]^2.$$

The minimum required sample size is the smallest integer value of n satisfying the inequality; with n known, the k -factors can be computed.

10.4 Gamma


Takagi (1972) considers the gamma random variable X with unknown location $\delta \leq x < \infty$ and estimated shape $\nu > 0$ and scale $\lambda > 0$ (scale) parameters, support $x \in [\delta, \infty)$, and density function:

$$f(x; \nu, \lambda, \delta) = \frac{\{(x - \delta) / \lambda\}^{\nu-1}}{\lambda \Gamma(\nu)} e^{-(x-\delta)/\lambda}.$$

The moments for this distribution are:

$$\begin{aligned} \mu &= \delta + \nu\lambda \\ \sigma^2 &= \nu\lambda^2 \\ \xi_3 &= 2/\sqrt{\nu} \\ \xi_4 &= 3 + 6/\nu. \end{aligned} \tag{Eq. 13}$$

With the same normal approximation $N(\mu_A, \sigma_A^2)$ for $A(k, n)$, the expansion factor given in equation (12) is computed using the gamma moments in equations (13). The derivation is essentially the same as that for the normal distribution, resulting in:

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$$n \geq e(k, \xi_3, \xi_4) \left[\frac{z_\alpha + z_\beta}{t_{p_1} - t_{p_0}} \right]^2 \quad (\text{Eq. 14})$$

and

$$k = - \left[\frac{z_\alpha t_{p_1} + z_\beta t_{p_0}}{z_\alpha + z_\beta} \right]^2. \quad (\text{Eq. 15})$$

Here,

$$t = \frac{X - \delta}{\nu}$$

is a standard gamma deviate with density:

$$f(t; \nu) = \frac{x^{\nu-1}}{\Gamma(\nu)} e^{-t}$$

for which $\mu_T = 0$ and $\delta_T = 1$.

The minimum required sample size is the smallest integer value of n satisfying the inequality; with n known, the k -factors can be computed.

10.5 Weibull


Takagi (1972) also considers the Weibull random variable X with unknown location $\delta \leq x < \infty$ and estimated shape $\nu > 0$ and scale $\lambda > 0$ (scale) parameters, support $x \in [\delta, \infty)$, and density function:

$$f(x; \nu, \lambda, \delta) = \frac{\nu}{\lambda} \left(\frac{x - \delta}{\lambda} \right)^{\nu-1} e^{[-(x-\delta)/\lambda]^\nu}.$$

The moments for this distribution are:

$$\begin{aligned} \mu &= \delta + \lambda \mu_T \\ \sigma^2 &= \lambda^2 \sigma_T^2 \\ \xi_3 &= \left[\Gamma(3b+1) - 3\Gamma(2b+1) + 2\Gamma^3(b+1) \right] / \sigma_T^3 \\ \xi_4 &= \left[\Gamma(4b+1) - 4\Gamma(3b+1)\Gamma(2b+1) + 6\Gamma^2(2b+1)\Gamma(b+1) \right] / \sigma_T^4. \end{aligned} \quad (\text{Eq. 16})$$

Here,

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$$T = \frac{(x - \gamma)}{\eta}$$

is a standard Weibull random variable with density:

$$f(t; \nu) = \nu t^{\nu-1} e^{-t^\nu}$$

and mean and variance:

$$\begin{aligned}\mu_T &= \Gamma(b+1) \\ \sigma_T^2 &= \Gamma(2b+1) - \Gamma^2(b+1)\end{aligned}$$

where $b = 1/\nu$.

The expansion factor in (Eq. 12) is computed using the Weibull moments in (Eq. 16). The expressions for n and k are again given by (Eq. 14) and (Eq. 15), where it is understood that now t is the Weibull deviate.

10.6 Inverse Gaussian

Aminzadeh (1996) considers the inverse-Gaussian random variable X with unknown location parameter $\delta \leq x < \infty$, estimated mean $\mu > 0$ and shape parameter $\lambda > 0$, support $x \in [0, \infty)$, and distribution function:


$$F(x; \mu, \lambda) = F_1(x; \mu, \lambda) + F_2(x; \mu, \lambda)$$

where:

$$\begin{aligned}F_1(x; \mu, \lambda) &= \varphi\left(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} - 1\right)\right) \\ F_2(x; \mu, \lambda) &= e^{2\lambda/\mu} \varphi\left(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} + 1\right)\right)\end{aligned}$$

and $\varphi(\cdot)$ is the distribution function of the standard normal distribution $N(0,1)$. As explained below, the derivation of the inverse-Gaussian sampling plan appears to be flawed.

As λ tends to infinity, the inverse-Gaussian distribution $F(x; \mu, \lambda)$ is asymptotically normal with distribution $F_1(x; \mu, \lambda)$. On this basis, it is argued that if the shape-to-mean ratio is large ($\lambda/\mu > 10$), then $F_2(x; \mu, \lambda)$ is almost zero the failure probability of $p = F_1(x; \mu, \lambda)$. This relationship is used to determine the means for the null and alternative distributions as described in Section 9.0.

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While it is true that for $F_2(x; \mu, \lambda)$ is almost zero when $\lambda/\mu > 10$, it is also true that $F_1(x; \mu, \lambda)$ is almost zero. Thus, the second term cannot simply be ignored in the calculation of p_0 and p_1 . This is illustrated in Figures 10.0-1 and 10.0-2 for $\lambda/\mu = 10$ and lower limit $x_{\min} \in [0, \mu]$. Figure 10.0-1 shows the values of $F_1(x; \mu, \lambda)$ and $F_2(x; \mu, \lambda)$ as functions of the limit-to-mean ratio $x_{\min}/\mu \in [0, 1]$. Also shown is the sum and ratio of these two terms, $p = F_1(x; \mu, \lambda) + F_2(x; \mu, \lambda)$ and $R = F_2(x; \mu, \lambda)/F_1(x; \mu, \lambda)$. Clearly, the contribution of $F_2(x; \mu, \lambda)$ in the computation of p is nontrivial for the entire range of lower limit. For the small failure probabilities typically of interest in acceptance sampling, the contribution of $F_2(x; \mu, \lambda)$ is nearly equal to that of $F_1(x; \mu, \lambda)$. Figure 10.0-2 illustrates the dependency of ratio R on the shape parameter for selected values of $\lambda/\mu \geq 10, 20$, and 100. Note that as λ/μ increases and the distribution becomes increasingly normal in shape, the contribution $F_2(x; \mu, \lambda)$ in the computation of p diminishes, but remains nontrivial, especially for small failure probabilities.

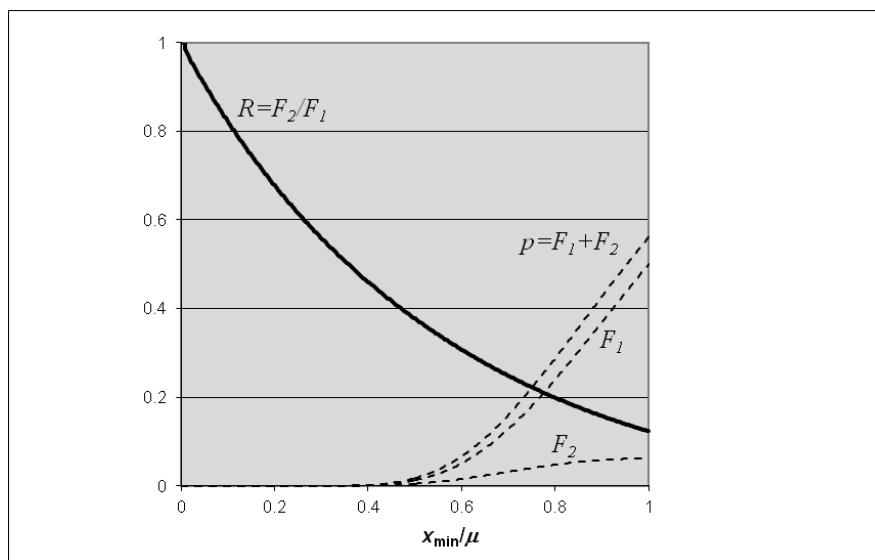



Figure 10.0-1. The Individual Terms of the Inverse-Gaussian Distribution, the Ratio $R = F_2(x; \mu, \lambda)/F_1(x; \mu, \lambda)$, and the Sum $p = F_1(x; \mu, \lambda) + F_2(x; \mu, \lambda)$, as Functions of the Lower-Limit-to-Mean Ratio x_{\min}/μ for the Shape-to-Mean Ratio $\lambda/\mu = 10$

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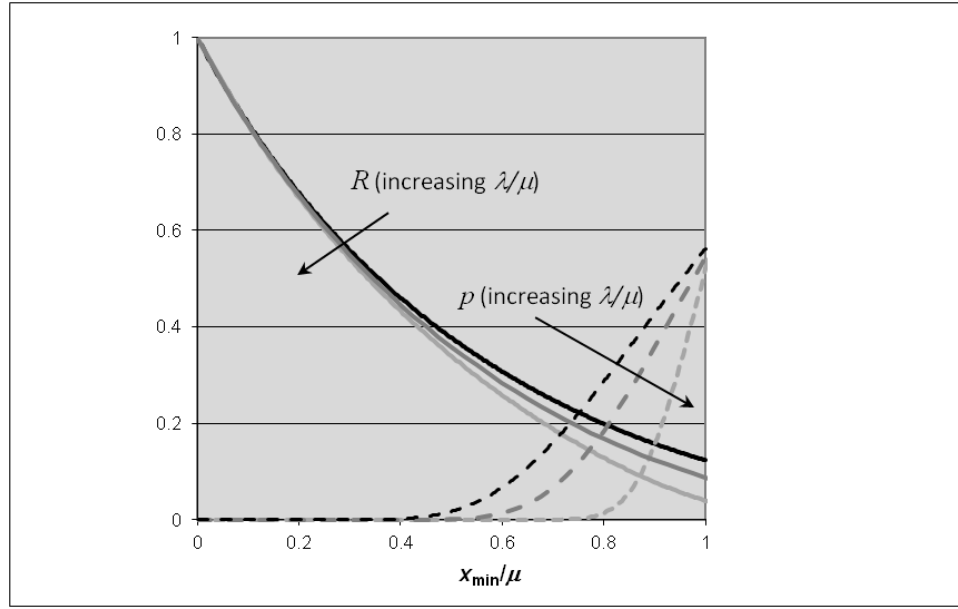


Figure 10.0-2. The Ratio of the Terms of the Inverse-Gaussian Distribution, $R = F_2(x; \mu, \lambda) / F_1(x; \mu, \lambda)$, and the Corresponding Failure Probability, p , as Functions of the Lower-Limit-to-Mean Ratio x_{\min}/μ , Parameterized on the Shape-to-Mean Ratio for $\lambda/\mu = 10, 20, 100$

10.7 Poisson


Guenther (1972, 1977) considers the Poisson random variable X with unknown mean μ , supports $x \in [0, 1, 2, \dots]$, and distribution function:

$$F(x; \mu) = e^{-\mu} \sum_{i=0}^x \frac{\mu^i}{i!},$$

The means μ_0 and μ_1 can be determined from the inverse distribution. Alternately, Guenther exploits the relationship between Poisson and chi-squared distributions and calculates the means using the expression:

$$\mu = \frac{\chi_{2x_{\min}; 1-p}^2}{2}.$$

Consider X as a discrete random variable representing the number of usable items in a lot and specification limit x_{\min} as the specified minimum number of usable items in each lot. If a sample of n lots are obtained, the total number of usable items obtained is the sum:

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$$Y = \sum_{j=0}^n X_j ,$$

which is distributed Poisson with a mean of $n\mu$, i.e., $F(y; n\mu)$. Because the team is working with discrete random variables (as is the case in attributes acceptance sampling), the rejection criterion is based on the minimum total number of usable items in all of the lots, $y \geq c + 1 = d$, instead of the distance k . Inequalities (Eq. 1) and (Eq. 2) are:

$$P_a(p \geq p_0) = 1 - F(y_d; n\mu_0) \geq 1 - \alpha$$

$$P_a(p \geq p_1) = 1 - F(d; n\mu_1) \leq \beta.$$

Or, more simply, the power requirements $F(d; n\mu_1) \geq 1 - \beta$ and $F(d; n\mu_0) \leq \alpha$. Again exploiting relationship between Poisson and chi-squared distributions, the power requirements imply:

$$\frac{\chi_{2d;\alpha}^2}{2\mu_0} \geq n \geq \frac{\chi_{2d;1-\beta}^2}{2\mu_1}. \quad (\text{Eq. 14})$$

This expression can be solved for a (d,n) sampling plan by enumeration. The value of integer values of d is increased until an integer value of n is found which satisfies both inequalities (Eq. 14).


11.0 Conclusions

The work reported in this paper is the first phase of a project intended to make ASV a practical alternative to ASA when appropriate variables plans are available. Published plans for exponential, normal, gamma, Weibull, and inverse-Gaussian random variables were discovered while the literature was reviewed. With the exception of normal plans, the search for off-the-shelf ASV plan calculators was fruitless.

To perfect an understanding and to facilitate the future development of variable plans for a wider selection of distributions, a consistent notation was introduced and interpreted the procedure for developing plans within the common framework of hypothesis testing (following the lead of Guenther (1972, 1977)). The result presented here is a consolidation of the existing literature.

In reviewing the methods presented, the mathematical derivations were verified. One unanticipated result was the discovery of a flaw in a published plan for inverse-Gaussian random variables that renders this plan unusable without modification. The flaw is subtle and this plan has remained unchallenged for over a decade.

As reported in Volume II (Probabilistic Requirements (Partial) Verification Methods Best Practices Improvement), the completion of this foundational research allowed team members to implement plan calculators and test the accuracy of these plans empirically. Testing is necessary since, as this Volume has shown, the majority of variables plans are based on approximating the

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standard deviation by employing an expansion factor to the sample standard deviation. The final phase of this project will address the practical application of ASV when distributional forms are themselves uncertain. The results of this additional research are reported in companion papers.

12.0 Acronyms List

ASA	Acceptance Sampling by Attributes
ASV	Acceptance Sampling by Variables
ATK	Alliant Techsystems, Inc.
CxP	Constellation Program
LaRC	Langley Research Center
MIL-STD	Military Standard
MSFC	Marshall Space Flight Center
MTSO	Management Technical Support Office
NESC	NASA Engineering and Safety Center
NRB	NESC Review Board
OC	Operating Characteristic

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14.0 Appendix

Appendix A. Acceptances Sampling by Attributes

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Appendix A. Acceptances Sampling by Attributes

A producer ships a large number of bolts built to withstand 500 foot-pounds of torque without breaking. The shipment comes in batches called lots, with 20,000 bolts/lot. Each lot will be inspected to determine if the bolts satisfy the minimum torque requirement. Inspection is destructive (a broken bolt cannot be used), so acceptance sampling must be applied. A sample of n bolts from each lot will be drawn at random and each bolt in the sample will be tested to see if it will withstand 500 foot-pounds of torque. If more than c bolts in the lot break, then the entire lot will be rejected. Otherwise, the entire lot will be accepted.


The requirement is:

- (1) *Performance indicator.* An individual bolt is conforming if $X \geq 500$ foot-pounds, where X is applied torque at which the bolt breaks during testing. Otherwise, the bolt is nonconforming.
- (2) *Minimum reliability.* A lot is conforming if no more than 100 bolts (0.5 percent) in that lot are nonconforming. That is, the minimum reliability required is $\rho = 0.995$. If p is the *failure probability* for the lot, then requirement is $p \leq 1 - \rho = 0.005$.
- (3) *Maximum acceptable consumer's risk.* To achieve the benefits of sampling, the consumer recognizes that occasionally a nonconforming lot will be accepted as conforming when it is not. The consumer requires that this happen on average no more than once for every twenty nonconforming received. That is, the maximum consumer's risk is $\beta = 0.05$.

How large should each sample be? How many individual nonconforming bolts can be observed in the sample and still accept the entire lot as conforming to the requirement?

Let $Y(n)$ be the number of bolts in a sample of n bolts that fail. Since each sampled bolt will be broken, the sampling in this case is sampling without replacement. The distribution of $Y(n)$ is therefore hypergeometric. However, if n is less than 10 percent of the lot size, then the binomial distribution is a good approximation. Assume therefore that $Y(n)$ will have a binomial distribution with the probability of failure p .

Figure 1 is a screen shot of the binomial (attributes) sampling plan calculator developed during this assessment. A minimum reliability of 0.995 and a consumer's risk of 0.05 have been entered as inputs. The third input is the reliability at which the producer's risk is calculated. The results shown provide twenty (n, c) sampling plans, parameterized on producer's risk. For example, a (1549, 4) plan is sufficient to test the requirement with an associated producer's risk of $\alpha = 0.0385$. That is, samples of 1549 bolts (7.774 percent of the 20,000 bolts in a lot) will be

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drawn at random for testing. Lots with samples which include more than four nonconforming bolts will be accepted. Lots with a failure probability greater than 0.005 will be accepted of 5 percent of the time on average. Lots with failure probability less than 0.999 will be rejected 3.85 percent on average. Also shown are the 20 corresponding OC curves, which plot the acceptance probability as a function of the maximum failure probability for each plan. Note, that by design, all of the OC curves intersect at $p = 0.005$ and $P_a = 0.05$.

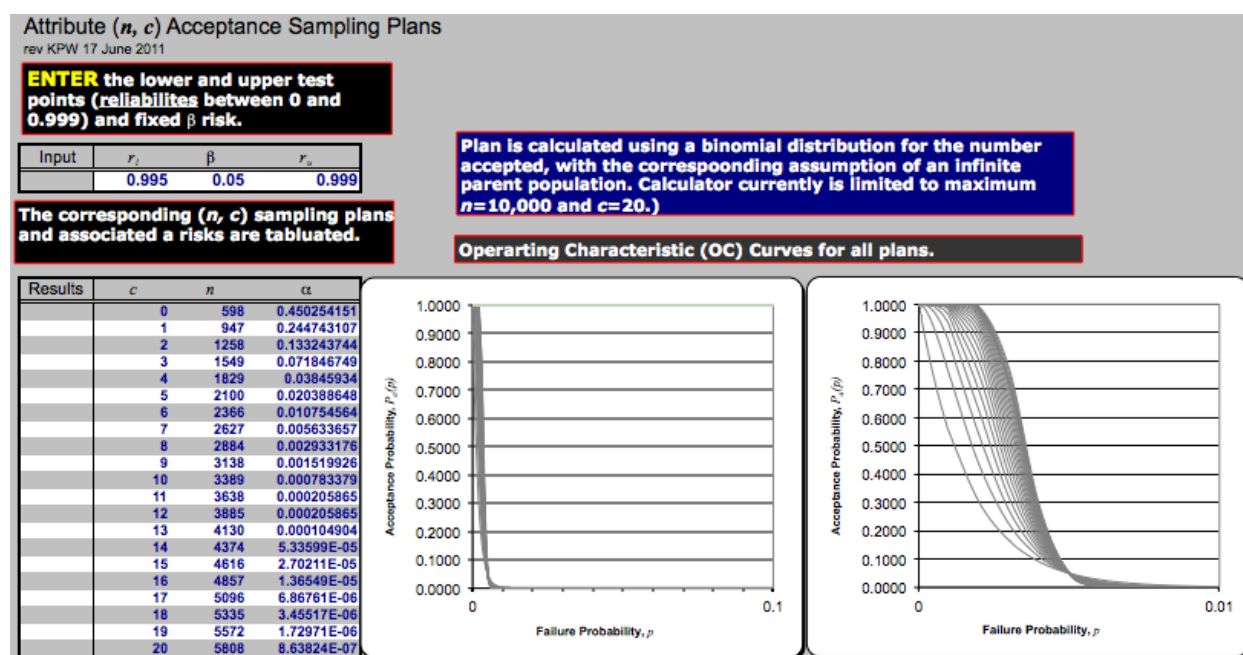



Figure 1. Screen Shot of the User Interface for the Binomial Sampling Plan Calculator

Acceptance Sampling by Variables (ASV)

The ASV plan requires destruction of a considerable fraction of the bolts in each lot (but, obviously, far less than 100-percent inspection). Suppose that, based on past experience, it can be assumed that the random breaking torques have a normal distribution with mean $\mu = 528$ and standard deviation $\sigma = 10$. If this assumption is can be sustained, then a sampling plan based on normal random variables should reduce the waste from inspection. Let the random variables X_1, X_2, \dots, X_n be a random sample n such measurements, with sample mean:

$$\bar{X}(n) = \frac{1}{n} \sum_{i=1}^n T_i$$

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and sample standard deviation:

$$S(n) = \sqrt{\frac{1}{n(n-1)} \sum_{i=0}^n (X_i - X(n))^2}.$$

How large should this sample be? How small can the sample mean be and still accept the entire lot as conforming to the requirement?

Figure 2 is a screen shot of the normal sampling plan calculator developed during this assessment. The historical mean and standard deviation have been entered as inputs. The required reliability, consumer’s risk, maximum reliability, and lower limit on torque also have been entered as inputs.

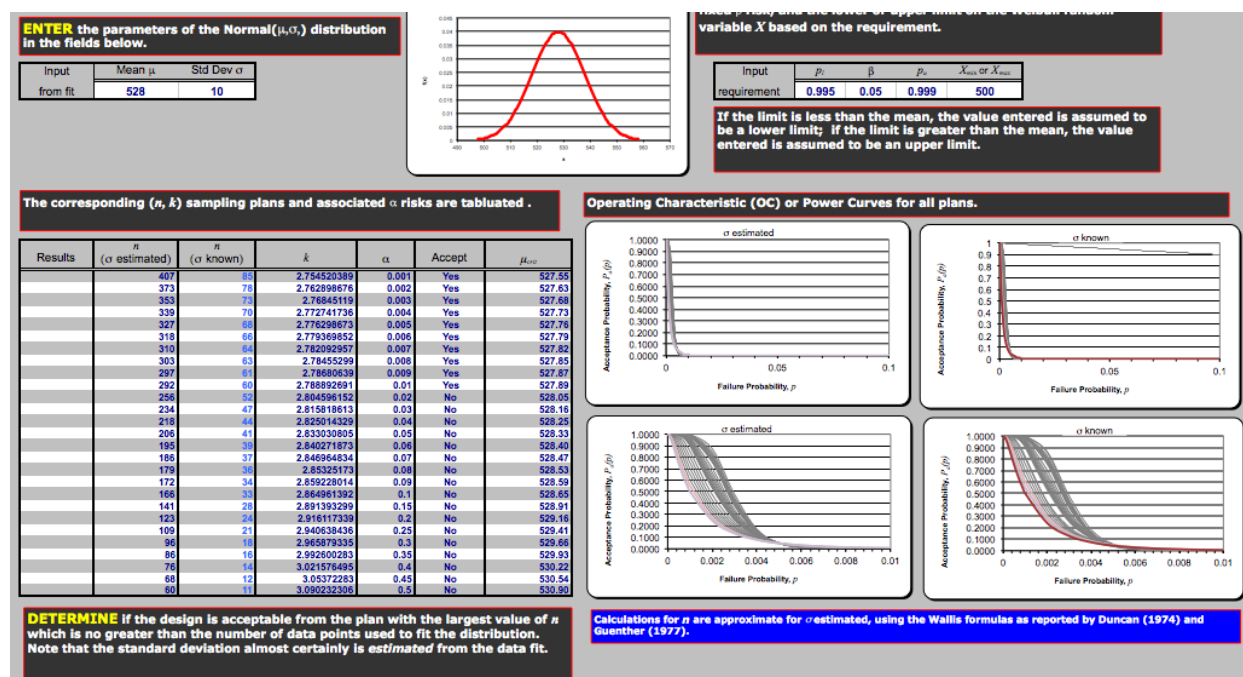



Figure 2. Screen Shot of the User Interface for the Normal Sampling Plan Calculator

The results shown provide 20 (n, k) sampling plans, parameterized on producer’s risk. For the same requirement, a $(218, 2.283)$ normal plan provides comparable protection to the $(1549, 4)$ binomial plan examined in preceding example, with a producer’s risk of $\alpha = 0.040$. Indeed, if the historical standard deviation is certain, then an even smaller $(44, 2.283)$ plan will suffice. The significant reduction in sample size is apparent.

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Also included in the results is a column which shows whether or not a normal distribution with given input parameters will be accepted (“Yes”) or not (“No”) under the corresponding plan. This is calculated from the inequality:

$$\mu - k\sigma \geq x_{min}$$

and the result indicates whether or not the inequality holds. A final column gives the critical value μ_{crit} , which is the value for which the equality holds exactly (i.e., the minimum of the mean that will result in acceptance).

The historical parameters for this problem were selected to illustrate an important cautionary. The *same* underlying distribution may lead to acceptance under a larger plan and rejection under a smaller plan. This is true in the example because the mean was chosen to be very close to the critical values for each of these plans. Note that for the (292, 2.7889) plan:

$$\mu - k\sigma = 500.12 > 500$$

and for the (256, 2.8045):

$$\mu - k\sigma = 499.96 < 500.$$

Depending on the application, one might question whether such a small difference (which is totally an artifact of the small difference in the k -value for the test) is meaningful. When this is the case, clearly larger samples on the whole will provide less conservative results.

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